Applied Math 2
Test #1

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1 Determine the order of the following expressions as $\epsilon \to 0$.

(i) $\sin^{-1} \frac{\epsilon}{\sqrt{1+\epsilon}}$

The following Maclaurin expansions will be used:

$$\sin^{-1}(x) = x + \frac{x^3}{6} + O(x^5), \qquad x \to 0$$

$$(1+x)^{-1/2} = 1 - \frac{x}{2} + O(x^2), \qquad x \to 0$$

Employing these expansions gives

$$\sin^{-1} \frac{\epsilon}{\sqrt{1+\epsilon}} = \epsilon \left[\left(1 - \frac{\epsilon}{2} + O(\epsilon^2)\right) + \frac{\left(1 - \frac{\epsilon}{2} + O(\epsilon^2)\right)^3}{6} + O(\epsilon^5) \right], \qquad \epsilon \to 0$$

$$= \left[\left(\epsilon - \frac{\epsilon^2}{2} + O(\epsilon^3)\right) + \frac{\left(\epsilon - \frac{3\epsilon^2}{2} + O(\epsilon^3)\right)}{6} + O(\epsilon^5) \right], \qquad \epsilon \to 0$$

$$= \frac{7\epsilon}{6} - \frac{3\epsilon^2}{4} + O(\epsilon^3), \qquad \epsilon \to 0$$

So finally

$$\boxed{\sin^{-1} \frac{\epsilon}{\sqrt{1+\epsilon}} = O(\epsilon), \qquad \epsilon \to 0}$$

To verify, compute the appropriate limit.

$$\lim_{\epsilon \to 0} \frac{\sin^{-1} \frac{\epsilon}{\sqrt{1+\epsilon}}}{\epsilon} = \frac{0}{0}$$

Use L'Hospital's rule.

$$= \lim_{\epsilon \to 0} \frac{\frac{2+\epsilon}{2(1+\epsilon)^{3/2}\sqrt{2-\epsilon-\frac{1}{1+\epsilon}}}}{1} = 1$$

(ii) $\ln\left(\sinh \frac{1}{\epsilon}\right)$

First rewrite the given function.

$$\ln\left(\sinh \frac{1}{\epsilon}\right) = \ln\left(\frac{1}{2}\left(e^{1/\epsilon} - e^{-1/\epsilon}\right)\right)$$

$$= \ln \frac{1}{2} + \ln\left(e^{1/\epsilon} - e^{-1/\epsilon}\right)$$

$$= \ln \frac{1}{2} + \ln\left(e^{1/\epsilon}\left(1 - e^{-2/\epsilon}\right)\right)$$

$$= \ln \frac{1}{2} + \ln e^{1/\epsilon} + \ln\left(1 - e^{-2/\epsilon}\right)$$

$$= \ln \frac{1}{2} + 1/\epsilon + \ln\left(1 - e^{-2/\epsilon}\right)$$

Since $e^{-2/\epsilon} \to 0$ as $\epsilon \to 0$, the log function can be expanded.

$$\ln(1-x) = -x - \frac{x^2}{2} + O(x^3), \qquad x \to 0$$

Now,

$$\ln\left(\sinh \frac{1}{\epsilon}\right) = \ln \frac{1}{2} + 1/\epsilon + \left(-e^{-2/\epsilon} - e^{-4/\epsilon} + \dots\right), \qquad \epsilon \to 0$$

Since each exponential function on the right hand side approaches 0 as $\epsilon \to 0$, the largest term is $1/\epsilon$, and so

$$\boxed{\ln\left(\sinh \frac{1}{\epsilon}\right) = O\left(\frac{1}{\epsilon}\right)}$$

Verify by computing the appropriate limit.

$$\lim_{\epsilon \to 0} \frac{\ln \frac{1}{2} + 1/\epsilon + \ln\left(1 - e^{-2/\epsilon}\right)}{1/\epsilon}$$

$$= \lim_{\epsilon \to 0} \epsilon \ln \frac{1}{2} + 1 + \epsilon \ln\left(1 - e^{-2/\epsilon}\right)$$

$$= \lim_{\epsilon \to 0} 1 + \epsilon \ln 1 = 1$$

(iii) $\ln\left[1+\ln\frac{1+2\epsilon}{\epsilon}\right]$

Rewrite the function.

$$\begin{aligned}\ln\left[1+\ln\frac{1+2\epsilon}{\epsilon}\right] &= \ln\left[(1-\ln\epsilon)+\ln(1+2\epsilon)\right]\\ &= \ln\left[(1-\ln\epsilon)\left(1+\frac{\ln(1+2\epsilon)}{1-\ln\epsilon}\right)\right]\\ &= \ln(1-\ln\epsilon)+\ln\left(1+\frac{\ln(1+2\epsilon)}{1-\ln\epsilon}\right)\end{aligned}$$

Now the second term in the log function approaches 0 as $\epsilon \to 0$, and so the log function approaches 0 as well. The biggest term is the first one, so

$$\boxed{\ln\left[1+\ln\frac{1+2\epsilon}{\epsilon}\right] = O(\ln(1-\ln\epsilon))}$$

Verify by finding the appropriate limit.

$$\lim_{\epsilon\to 0}\frac{\ln\left[1+\ln\frac{1+2\epsilon}{\epsilon}\right]}{\ln(1-\ln\epsilon)} = \frac{\infty}{\infty}$$

Use L'Hospital's rule.

$$= \lim_{\epsilon\to 0}\frac{1-\ln\epsilon}{(1+2\epsilon)(1+\ln(2+1/\epsilon))} = \frac{\infty}{\infty}$$

Use L'Hospital's rule.

$$= \lim_{\epsilon\to 0}\frac{-1/\epsilon}{2-1/\epsilon+2\ln(2+1/\epsilon)} = \frac{\infty}{\infty}$$

Use L'Hospital's rule.

$$= \lim_{\epsilon\to 0}\frac{1/\epsilon^2}{1/(\epsilon^2+2\epsilon^3)} = \lim_{\epsilon\to 0}\frac{\epsilon^2+2\epsilon^3}{\epsilon^2} = \lim_{\epsilon\to 0}1+2\epsilon = 1$$

(iv) $\ln\left[1+\frac{\ln\frac{1+2\epsilon}{\epsilon}}{1-2\epsilon}\right]$

Rewrite the expression using log rules.

$$\ln\left[1+\frac{1}{1-2\epsilon}\left(\ln(1+2\epsilon)-\ln\epsilon\right)\right]$$

Rearrange terms.

$$\ln\left[\left(1-\frac{\ln\epsilon}{1-2\epsilon}\right)+\frac{\ln(1+2\epsilon)}{1-2\epsilon}\right]$$

Factor the inside.

$$\ln\left[\left(1-\frac{\ln\epsilon}{1-2\epsilon}\right)\left(1+\frac{\ln(1+2\epsilon)}{1-2\epsilon}\left(\frac{1}{1-\frac{\ln\epsilon}{1-2\epsilon}}\right)\right)\right]$$

Use the log rule one more time.

$$\ln\left(1-\frac{\ln\epsilon}{1-2\epsilon}\right)+\ln\left(1+\frac{\ln(1+2\epsilon)}{1-2\epsilon}\left(\frac{1}{1-\frac{\ln\epsilon}{1-2\epsilon}}\right)\right)$$

Now the ln() on the right is in the form $\ln(1+x), x \to 0$, so that term also goes to zero with ϵ. Therefore

$$\boxed{\ln\left[1+\frac{\ln\frac{1+2\epsilon}{\epsilon}}{1-2\epsilon}\right]=O\left(\ln\left(1-\frac{\ln\epsilon}{1-2\epsilon}\right)\right)}$$

The solution can be verified with Mathematica:

```
Limit[(Log[1+Log[(1+2ϵ)/ϵ]/(1-2ϵ)])/(Log[1-Log[ϵ]/(1-2ϵ)]),ϵ → 0]
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2 For small ϵ, determine two terms in the expansion of each root of the following equations:

(i) $x^3-(3+\epsilon)x-2+\epsilon=0$

Let $x=x_0+\epsilon x_1+\epsilon^2x_2+\ldots$.

$$(x_0+\epsilon x_1+\epsilon^2x_2+\ldots)^3-(3+\epsilon)(x_0+\epsilon x_1+\epsilon^2x_2+\ldots)-2+\epsilon=0$$

The $O(1)$ terms give the equation:

$$x_0^3-3x_0-2=0 \qquad \Rightarrow x_0=-1,-1,2$$

The double root -1 will be dealt with after the expansion of the root 2 is found.

The $O(\epsilon)$ terms give the equation:

$$3x_0^2x_1-3x_1-x_0+1=0$$

For $x_0=2$, this equation becomes

$$12x_1 - 3x_1 - 2 + 1 = 0 \qquad \Rightarrow x_1 = \frac{1}{9}$$

So for the root 2, the expansion is $\boxed{x = 2 + \frac{1}{9}\epsilon + \ldots}$.

Now let $x = -1 + \lambda$. This leads to the equation

$$\lambda^3 - 3\lambda^2 - \epsilon\lambda + 2\epsilon = 0$$

For small λ and $\epsilon \to 0$, the dominant terms are λ^2 and ϵ, so let $\lambda = O(\sqrt{\epsilon})$ and establish that half integer powers of epsilon should occur in the asymptotic series used to expand x. Therefore let

$$x = -1 + \epsilon^{1/2}x_1 + \epsilon x_2 + \ldots$$

$$(-1 + \epsilon^{1/2}x_1 + \epsilon x_2 + \ldots)^3 - (3 + \epsilon)(-1 + \epsilon^{1/2}x_1 + \epsilon x_2 + \ldots) - 2 + \epsilon = 0$$

The $O(1)$ terms give the identity $-1+3-2 = 0$. The $O(\epsilon^{1/2})$ terms give the identity $3x_1 - 3x_1 = 0$. The $O(\epsilon)$ terms give the equation $-3x_1^2 + 2 = 0$ which implies $x_1 = \pm\sqrt{\frac{2}{3}}$.

So the two expansions for this root are

$$\boxed{x = -1 + \sqrt{\frac{2}{3}}\epsilon + \ldots}$$

and

$$\boxed{x = -1 - \sqrt{\frac{2}{3}}\epsilon + \ldots}$$

This can be verified with the Mathematica code:

```
roots = Solve[x³ − (3 + ϵ)x − 2 + ϵ == 0, x];
Series[x /.  roots, ϵ, 0, 1] // FullSimplify
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(ii) $\epsilon u^3 + (u-2)^2 = 0$

The expanded equation is $\epsilon u^3 + u^2 - 4u + 4 = 0$. Let $u = u_0 + \epsilon u_1 + O(\epsilon^2)$. Plug back in the expanded equation.

$$\epsilon(u_0^3 + O(\epsilon)) + (u_0^2 + \epsilon(2u_0u_1) + O(\epsilon^2)) - 4(u_0 + \epsilon u_1 + O(\epsilon^2)) + 4 = 0$$

$$u_0^2 - 4u_0 + 4 + \epsilon(u_0^3 + 2u_0u_1 - 4u_1) + O(\epsilon^2) = 0$$

This equation implies

$$u_0^2 - 4u_0 + 4 = 0 \rightarrow u_0 = 2, 2$$

$$u_0^3 + 2u_0u_1 - 4u_1 = 0 \rightarrow 8 + 4u_1 - 4u_1 \neq 0$$

The contradiction in the last equation demonstrates that this is not the correct expansion to use. To find the expansions of the double root, set $u = 2+\lambda$ and plug into the original equation.

$$\epsilon(2+\lambda)^3 + (2+\lambda-2)^2 = 0$$

$$\epsilon(8 + 12\lambda + 6\lambda^2 + \lambda^3) + \lambda^2 = 0$$

For small λ and $\epsilon \rightarrow 0$, the dominant terms are λ^2 and ϵ, so set $\lambda = O(\sqrt{\epsilon})$ and expand u around 2 in half integer powers of ϵ.

$$u = 2 + u_1\sqrt{\epsilon} + u_2\epsilon + O(\epsilon^{3/2})$$

$$\epsilon(2 + u_1\sqrt{\epsilon} + O(\epsilon))^3 + (2 + u_1\sqrt{\epsilon} + u_2\epsilon + O(\epsilon^{3/2}) - 2)^2 = 0$$

$$\epsilon(8 + 12u_1\sqrt{\epsilon} + O(\epsilon)) + u_1^2\epsilon + O(\epsilon^{3/2}) = 0$$

Divide by ϵ.

$$8 + u_1^2 + (2u_1u_2 + 12u_1)\sqrt{\epsilon} + O(\epsilon) = 0$$

This equation implies $8 + u_1^2 = 0 \rightarrow u_1 = \pm i2\sqrt{2}$. So the expansions of the double root are

$$\boxed{u = 2 + i2\sqrt{2}\sqrt{\epsilon} + O(\epsilon)}$$

$$\boxed{u = 2 - i2\sqrt{2}\sqrt{\epsilon} + O(\epsilon)}$$

Now let $u = \bar{u}\epsilon^{-p}$ and plug into the original expanded equation.

$$\bar{u}^3 + \bar{u}\epsilon^{p-1} - 4\bar{u}\epsilon^{2p-1} + 4\epsilon^{3p-1} = 0$$

Balance the first two terms by setting $p = 1$.

$$\bar{u}^3 + \bar{u} - 4\bar{u}\epsilon + 4\epsilon^2 = 0$$

Now proceed with a regular expansion of $\bar{u}$.

$$\bar{u} = \bar{u_0} + \bar{u_1}\epsilon + \bar{u_2}\epsilon^2 + \dots$$

This leads to the equation

$$\bar{u_0}^3 + \bar{u_0}^2 + \epsilon(3\bar{u_0}^2\bar{u_1} + 2\bar{u_0}\bar{u_1} - 4\bar{u_0}) + O(\epsilon^2) = 0$$

This equation implies

$$\bar{u_0}^3 + \bar{u_0}^2 = 0 \rightarrow \bar{u_0} = 0, 0, -1$$

Choose $\bar{u_0} = -1$ and use in the next equation.

$$3\bar{u_0}^2\bar{u_1} + 2\bar{u_0}\bar{u_1} - 4\bar{u_0} = \bar{u_1} + 4 = 0 \rightarrow \bar{u_1} = -4$$

So the expansion in terms of $\bar{u}$ is

$$\bar{u} = -1 - 4\epsilon + O(\epsilon^2)$$

and the final expansion of the third root is

$$\boxed{u = \frac{-1}{\epsilon} - 4 + O(\epsilon)}$$

These solutions can be verified with Mathematica:

```
roots = Solve[ϵx^3 + (x − 2)^2 == 0, x];
Series[x /.  roots, ϵ, 0, 1] // FullSimplify
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3 Show that as $\epsilon \rightarrow 0$

(i) $\int_0^1 \frac{\sin \epsilon t}{t} \mathrm{d}t \sim \epsilon - \frac{1}{18}\epsilon^3 + \frac{1}{600}\epsilon^5$

Expand the sine function.

$$\int_0^1 \frac{\sin \epsilon t}{t} \mathrm{d}t = \int_0^1 \frac{1}{t}\left(\epsilon t - \frac{\epsilon^3 t^3}{3!} + \frac{\epsilon^5 t^5}{5!} + \dots\right) \mathrm{d}t$$

$$= \int_0^1 \left(\epsilon - \frac{\epsilon^3 t^2}{3!} + \frac{\epsilon^5 t^4}{5!} + \dots\right) \mathrm{d}t$$

Integrate.

$$= \epsilon - \frac{\epsilon^3}{3!3} + \frac{\epsilon^5}{5!5} + \dots$$

Finally,

$$\boxed{\int_0^1 \frac{\sin \epsilon t}{t} \mathrm{d}t = \epsilon - \frac{1}{18}\epsilon^3 + \frac{1}{600}\epsilon^5 + \ldots}$$

(ii) $\int_0^\epsilon t^{-3/4} e^{-t} \mathrm{d}t \sim 4\epsilon^{1/4} - \frac{4}{5}\epsilon^{5/4} + \frac{2}{9}\epsilon^{9/4}$

Expand the exponential function.

$$\int_0^\epsilon t^{-3/4} e^{-t} \mathrm{d}t = \int_0^\epsilon t^{-3/4} \left(1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \ldots\right) \mathrm{d}t$$

$$= \int_0^\epsilon t^{-3/4} - t^{1/4} + \frac{1}{2}t^{5/4} - \frac{1}{3!}t^{9/4} + \ldots \mathrm{d}t$$

$$\boxed{= 4\epsilon^{1/4} - \frac{4}{5}\epsilon^{5/4} + \frac{2}{9}\epsilon^{9/4} + \ldots}$$

4 Using regular perturbation expansion, solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} + y = \epsilon y^2, t > 0$$

$$t = 0 : y = 1$$

Let $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots$, and expand the IC as $y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) + \ldots = 1$. Now,

$$(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots)' + (y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots) = \epsilon(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots)^2$$

The $O(1)$ terms give the IVP

$$y_0' + y_0 = 0, \qquad y_0(0) = 1$$

The solution is $y_0(t) = e^{-t}$.

The $O(\epsilon)$ terms give the IVP

$$y_1' + y_1 = y_0^2 = e^{-2t}, \qquad y_1(0) = 0$$

The homogeneous solution is $y_{1h}(t) = c_1 e^{-t}$ and the particular solution is $y_{1p}(t) = c_1 e^{-t} \int_0^t \frac{1}{c_1} e^s e^{-2s} \mathrm{d}s = e^{-t} - e^{-2t}$, and so the general solution is $y_1(t) = c_2 e^{-t} - e^{-2t}$. The IC gives $y_1(0) = c_2 - 1 = 0 \Rightarrow c_2 = 1$, so the final solution is $y_1(t) = e^{-t} - e^{-2t}$.

Therefore a perturbation solution to this IVP is

$$\boxed{y(t) = e^{-t} + \epsilon(e^{-t} - e^{-2t}) + O(\epsilon^2)}$$

5 Consider the equation $\ddot{u} + \omega_0^2 u = \epsilon \dot{u}^2 u, \epsilon << 1$

(i) Determine a two term straight forward expansion and discuss its uniformity.

Let $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots$. Now the DE becomes

$$\begin{aligned}&(\ddot{u_0} + \epsilon\ddot{u_1} + \epsilon^2\ddot{u_2} + \ldots) + \omega_0^2(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots)\\ &= \epsilon(\dot{u}_0 + \epsilon\dot{u}_1 + \epsilon^2\dot{u}_2 + \ldots)^2(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots)\end{aligned}$$

The $O(1)$ terms give the DE $\ddot{u_0} + \omega_0^2 u_0 = 0$ with solution

$$\boxed{u_0(t) = A\cos\omega_0 t + B\sin\omega_0 t}$$

The $O(\epsilon)$ terms give the DE $\ddot{u_1} + \omega^2 u_1 = \dot{u_0}^2 u_0$. The solution is

$$\begin{aligned}u_1(t) = &\frac{1}{32}t\left(-4B\left(A^2+B^2\right)\omega_0\cos(t\omega_0) + 4A\left(A^2+B^2\right)\omega_0\sin(t\omega_0)\right)\\ &+\frac{1}{32}\left(2(A^3+AB^2+16c_1)\cos(t\omega_0) + A(A^2-3B^2)\cos(3t\omega_0)\right)\\ &+\frac{1}{32}\left(2(A^2B+B^3+16c_2)\sin(t\omega_0) - B(-3A^2+B^2)\sin(3t\omega_0)\right)\end{aligned}$$

The t multiplying the first part of the solution will make the straight-forward expansion solution nonuniform when $\boxed{t = O(1/\epsilon) \text{ as } \epsilon \to 0}$.

(ii) Determine a first-order uniform expansion by using the Lindstedt-Poincaré technique.

Let $\tau = (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots)t$, so $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}\tau}(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots)$.

Let $u = u_0 + \epsilon u_1 + \ldots$. Let $' = \frac{\mathrm{d}}{\mathrm{d}\tau}$. Now the DE becomes

$$\begin{aligned}&(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots)^2(u_0'' + \epsilon u_1'' + \ldots) + \omega_0^2(u_0 + \epsilon u_1 + \ldots)\\ &= \epsilon(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots)^2(u_0' + \epsilon u_1' + \ldots)^2(u_0 + \epsilon u_1 + \ldots)\end{aligned}$$

The $O(1)$ terms give the DE $u_0'' + \omega_0^2 u_0 = 0$ with solution $u_0(t) = A\cos\omega_0 t + B\sin\omega_0 t$. The $O(\epsilon)$ terms give the DE $u_1'' + \omega^2 u_1 = u_0'^2 u_0 - 2\omega_1 u_0''$. The RHS is

$$\begin{aligned}&\frac{1}{2}\omega_0^2(A\cos(\tau\omega_0) + B\sin(\tau\omega_0))\cdot\\ &\left(A^2 + B^2 + 4\omega_1 - (A^2 - B^2)\cos(2\tau\omega_0) + 2AB\sin(2\tau\omega_0)\right)\end{aligned}$$

To eliminate the source of secular terms, set

$$A^2 + B^2 + 4\omega_1 = 0$$

So that $\omega_1 = -\frac{1}{4}(A^2 + B^2)$. Now the first order ODE becomes

$$u_1'' + \omega^2 u_1 = \frac{1}{2}\omega_0^2 (A\cos(\tau\omega_0) + B\sin(\tau\omega_0)) \cdot \left(2AB\sin(2\tau\omega_0) - (A^2 - B^2)\cos(2\tau\omega_0)\right)$$

which has the solution $u_1(\tau) =$

$$\frac{1}{32}\left(2\left((A^2 + B^2)(A - 2B\tau\omega_0) + 16c_1\right)\cos(\tau\omega_0) + A(A^2 - 3B^2)\cos(3\tau\omega_0)\right)$$

$$+\frac{1}{32}\left(2\left((A^2 + B^2)(B + 2A\tau\omega_0) + 16c_2\right)\sin(\tau\omega_0) + B(-3A^2 + B^2)\sin(3\tau\omega_0)\right)$$

Now the first order uniform expansion is

$$\boxed{u(\tau) = u_0(\tau) + \epsilon u_1(\tau) + \ldots}$$

where u_0 and u_1 are defined above and $\tau = t(1 - \epsilon\frac{1}{4}(A^2 + B^2) + \ldots)$.